

THE PRINCIPAL ELEMENT OF A FROBENIUS LIE ALGEBRA

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ABSTRACT. We introduce the notion of the *principal element* of a Frobenius Lie algebra \mathfrak{f} . The principal element corresponds to a choice of $F \in \mathfrak{f}^*$ such that $F[-, -]$ non-degenerate. In many natural instances, the principal element is shown to be semisimple, and when associated to \mathfrak{sl}_n , its eigenvalues are integers and are independent of F . For certain “small” functionals F , a simple construction is given which readily yields the principal element. When applied to the first maximal parabolic subalgebra of \mathfrak{sl}_n , the principal element coincides with semisimple element of the principal three-dimensional subalgebra. We also show that Frobenius algebras are stable under deformation.

1. INTRODUCTION

In this note all Lie algebras will be complex but much of what is said holds more generally and probably even for most finite characteristics. Let \mathfrak{f} be a finite dimensional Lie algebra. Each linear functional $F \in \mathfrak{f}^*$ determines a skew-symmetric bilinear form B_F on \mathfrak{f} defined by $B_F(x, y) = F([x, y])$ for $x, y \in \mathfrak{f}$. The *index* of \mathfrak{f} is the minimum dimension of the kernel of B_F as F ranges over \mathfrak{f}^* . The Lie algebra \mathfrak{f} is *Frobenius* if its index is zero, i.e. if there exists an $F \in \mathfrak{f}^*$ such that B_F is non-degenerate. Such F are called *Frobenius functionals*.

The index of a Lie algebra has applications to invariant theory and has been extensively studied in that context. Frobenius Lie algebras also are of interest in deformation and quantum group theory. Specifically, if M is the matrix of the non-degenerate form B_F relative to some basis x_1, \dots, x_m of \mathfrak{f} then $R = \sum_{i,j} (M^{-1})_{ij} x_i \wedge x_j$ is a constant solution to the classical Yang-Baxter equation.

A simple Lie algebra itself can never be Frobenius but many subalgebras are. For \mathfrak{sl}_n , where we may assume the simple roots numbered from 1 to $n - 1$, basic results of Elashvili [4], [2] assert in particular

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that the i th maximal parabolic subalgebra (obtained by deleting the i th negative root) is Frobenius if and only if i and n are relatively prime. It is not known which non-maximal parabolic subalgebras are Frobenius. There are, however, algorithms for computing the index of the “seaweed” Lie algebras, i.e., one obtained by omitting some positive and some negative roots, and thus to check whether it is Frobenius. The suggestive name comes from their shape as pictured in [1], but the definition is meaningful for all simple Lie algebras. Seaweed algebras are called “biparabolic” by A. Joseph [8]. In particular, each parabolic subalgebra is also biparabolic. For \mathfrak{sl}_n , Dergachev and Kirillov [1] have given a simple algorithm in terms of “meander” graphs to compute the index of a seaweed Lie algebra. An alternative approach which works also for the biparabolic subalgebras of \mathfrak{sp}_n was obtained by Panyushev in [13].

For any skew bilinear form B on a vector space V one can define a linear map $V \rightarrow V^*$ sending $v \in V$ to the functional sending x to $B(v, x)$ for all $x \in V$. If B is non-degenerate then this map is an isomorphism. Thus, when \mathfrak{f} is Frobenius with functional F , there is a unique $\hat{F} \in \mathfrak{f}$ such that $F(x) = F([\hat{F}, x])$ for all $x \in \mathfrak{f}$. Our primary concern in this note is F , the **principal element** of \mathfrak{f} . The main results are:

- If a Frobenius Lie algebra \mathfrak{f} is a subalgebra of a simple Lie algebra \mathfrak{g} and if \mathfrak{f} is not an ideal of any larger subalgebra of \mathfrak{g} (a condition satisfied in particular when \mathfrak{f} contains a Cartan subalgebra of \mathfrak{g}) then the principal element is semisimple; we will call such a Frobenius subalgebra *saturated*.
- If, moreover, \mathfrak{g} is \mathfrak{sl}_n then the eigenvalues of $\text{ad } \hat{F}$ are integers which, together with their multiplicities, are independent of the choice of Frobenius functional.
- In some important special cases where \mathfrak{f} has a “small” Frobenius functional, in particular for the seaweed subalgebras of \mathfrak{sl}_n , we give a simple method for the computation of the principal element.
- Frobenius Lie algebras are stable under deformation.

The second result above probably holds for all simple \mathfrak{g} although it is possible (but seems unlikely) that in some cases the eigenvalues may be half-integers or even multiples of $1/3$. It will be seen that the principal element of the first parabolic subalgebra of \mathfrak{sl}_n is (up to scalar multiple) the semisimple element of Kostant’s principal three-dimensional subalgebra of \mathfrak{sl}_n , [9]. The principal three-dimensional

subalgebra is defined for all simple \mathfrak{g} ; we do not know if its semisimple element is always the principal element of some parabolic subalgebra.

This work arose as an attempt to generalize a result of [5] concerning the first maximal parabolic subalgebra of \mathfrak{sl}_n . The aforementioned result of Elashvili implies that this Lie algebra is Frobenius for any n since 1 and n are relatively prime. Elashvili's argument, however, does not provide any explicit Frobenius functionals. In [5] a canonical Frobenius functional was exhibited for the first maximal parabolic subalgebra of \mathfrak{sl}_n . In the preprint [6], we provide, among other results, a natural generalization of the results of [5] which is valid for all of the i th maximal parabolic subalgebras of \mathfrak{sl}_n with i relatively prime to n . The results of [6] make significant use of the principal elements of these algebras.

2. SEMISIMPLICITY OF THE PRINCIPAL ELEMENT

Theorem 1. *Suppose that \mathfrak{f} is a saturated Frobenius subalgebra of a simple Lie algebra \mathfrak{g} . If F is a Frobenius functional on \mathfrak{f} then its principal element \widehat{F} is semisimple.*

PROOF. Viewing \widehat{F} as an element of \mathfrak{g} it has a Jordan-Chevalley decomposition $\widehat{F} = \widehat{F}_s + \widehat{F}_n$ into semisimple and nilpotent parts which in any linear representation of \mathfrak{g} would be the same as its decomposition as a matrix (cf. [7, p. 24]) so we may assume that \mathfrak{g} is linear. There are then polynomials p_s and p_n , each without constant term, such that $p_s(T) + p_n(T) = T$ (T being any variable) and $\widehat{F}_s = p_s(\widehat{F})$, $\widehat{F}_n = p_n(\widehat{F})$. One has, further that $\text{ad}(\widehat{F})_s = \text{ad}(\widehat{F}_s)$ (which therefore may be written unambiguously as $\text{ad} \widehat{F}_s$) and similarly for \widehat{F}_n (cf. [7]). It follows that $\text{ad} \widehat{F}_s$ is a polynomial in $\text{ad} \widehat{F}$ and therefore that $[\widehat{F}_s, \mathfrak{f}] \subseteq \mathfrak{f}$. By hypothesis, this is possible only if \widehat{F}_s is already in \mathfrak{f} . By definition, $F \circ \text{ad} \widehat{F} = F$, so $F \circ \text{ad} \widehat{F}_s = p_s(1)F$ and similarly for \widehat{F}_n . It follows that $p_n(1) = 0$, else $\text{ad} \widehat{F}_n$, which is nilpotent, would have a non-zero eigenvalue. Since $p_s(1) + p_n(1) = 1$ it follows that $p_n(1) = 0$ so $F \circ \text{ad} \widehat{F}_s = F$. But \widehat{F} is the unique element of \mathfrak{g} with that property, so $\widehat{F} = \widehat{F}_s$. \square

As mentioned, the hypothesis that \mathfrak{f} is saturated, i.e., is not an ideal of any larger subalgebra of \mathfrak{g} , will be satisfied if \mathfrak{f} contains a Cartan subalgebra \mathfrak{h} of \mathfrak{g} .

3. INTEGRALITY AND INVARIANCE OF THE EIGENVALUES OF \widehat{F}

The semisimplicity of the principal element \widehat{F} implies that \mathfrak{f} can be decomposed according to the eigenvalues of \widehat{F} , $\mathfrak{f} = \bigoplus_{\lambda} \mathfrak{f}_{\lambda}$, where λ ranges over the eigenvalues of \widehat{F} and \mathfrak{f}_{λ} is the corresponding eigenspace. Since \widehat{F} is a derivation one has $[\mathfrak{f}_{\lambda}, \mathfrak{f}_{\mu}] \subseteq \mathfrak{f}_{\lambda+\mu}$.

Lemma 1. *One has $F(\mathfrak{f}_{\lambda}) = 0$ for $\lambda \neq 1$.*

PROOF. For $x \in \mathfrak{f}_{\lambda}$ we have $F(x) = F([\widehat{F}, x]) = F(\lambda x) = \lambda F(x)$. \square

Corollary 1. *The eigenspace \mathfrak{f}_{λ} is dual to $\mathfrak{f}_{1-\lambda}$ under B_F ; in particular \mathfrak{f}_0 is dual to \mathfrak{f}_1 .* \square

When $\mathfrak{f} \subset \mathfrak{sl}_n$ we can say more since the Cartan subalgebra of \mathfrak{sl}_n is rather elementary to work with.

Theorem 2. *The eigenvalues of \widehat{F} are integers which constitute a single unbroken string. That is, if i and j are eigenvalues with $i < j$, then any integer k with $i < k < j$ is also an eigenvalue.*

PROOF. Since \widehat{F} is semisimple it is contained in a Cartan subalgebra which we may assume to be the set of diagonal matrices of \mathfrak{sl}_n . Writing $\widehat{F} = \text{diag}(\lambda_1, \dots, \lambda_n)$, each eigenvalue of \widehat{F} is a difference $\lambda_i - \lambda_j$. Construct a (directed) graph whose vertices are the integers $\{1, \dots, n\}$ by connecting i to j whenever $\lambda_i - \lambda_j = 1$. Since 1 is an eigenvalue of \widehat{F} the assertion is that this graph is connected. If not, then we can write $\{1, \dots, n\}$ as a disjoint union $S \amalg S'$ where S is the set of vertices of any maximal connected subgraph and S' is its (non-empty) complement. Let $\#S = m$. Then the diagonal matrix $h = \text{diag}(\mu_1, \dots, \mu_n)$ with $\mu_i = 1/m$ for $i \in S$ and $\mu_i = -1/(m-n)$ for $i \in S'$ is in \mathfrak{sl}_n but one has $[h, x] = 0$ for all $x \in \mathfrak{f}$. If h is not in \mathfrak{f} then the hypothesis that \mathfrak{f} is saturated is violated and if it is in \mathfrak{f} then \mathfrak{f} is not Frobenius, so the graph is indeed connected. \square

The foregoing allows an alternate proof of a result of Ooms.

Theorem 3. *The eigenvalues of \widehat{F} , together with their multiplicities, depend only on \mathfrak{f} and not on the choice of a Frobenius functional F .*

PROOF. Those elements of \mathfrak{f}^* which are not Frobenius form a subvariety of \mathfrak{f}^* which, since the ground field is \mathbb{C} , is of codimension at least 2, so the set of Frobenius functionals is a connected subset of \mathfrak{f}^* . The eigenvalues of \widehat{F} are continuous functions of F as F varies in the set of Frobenius functionals, but being constrained to be integers they must be constants. \square

In view of the theorem one may speak simply of the eigenvalues of \mathfrak{f} , but the decomposition of \mathfrak{f} into eigenspaces of the principal element depends on the choice of a Frobenius functional. Given one, define \mathfrak{f}_+ to be the space spanned by those eigenspaces \mathfrak{f}_m with $m > 0$, and similarly for \mathfrak{f}_- . These are both modules over \mathfrak{f}_0 , which is generally not Abelian, and their dimensions generally differ since \mathfrak{f}_m is not dual to \mathfrak{f}_{-m} but to \mathfrak{f}_{1-m} .

Dergachev (private communication, hopefully to be posted) has generalized the foregoing and given examples to show that saturated Frobenius subalgebras are not determined up to conjugacy by their associated eigenvalues. These examples, however, form discrete rather than continuous families which suggests that up to conjugation there may nevertheless be only a finite number of saturated Frobenius subalgebras of \mathfrak{sl}_n .

4. DEFORMATIONS

For a finite dimensional Lie algebra \mathfrak{f} which is not necessarily Frobenius, call $F \in \mathfrak{f}^*$ an *index functional* if the index of B_F is that of \mathfrak{f} , i.e., the least possible.

Theorem 4. *The index of a finite dimensional Lie algebra does not increase under deformation.*

PROOF. Any vector space V with a skew form B can always be written as a direct sum $V = V_1 + V_0$ where V_0 is the kernel of B and V_1 is non-singular with a basis $x_1, \dots, x_m, y_1, \dots, y_m$ where $B(x_i, y_j) = \delta_{ij}$. So letting the algebra be L , of dimension n and index r , and F be an index functional on L , and setting $m = (n - r)/2$ there are $x_1, \dots, x_m, y_1, \dots, y_m \in L$ such that $B_F([x_i, y_j]) = \delta_{ij}$. A small change (or one with a formal parameter) in the Lie product leaves the matrix $B_F([x_i, y_j])$ non-singular so the subspace spanned by the x_i and y_j remains non-singular. \square

Corollary 2. *A finite dimensional Frobenius Lie algebra remains Frobenius under deformation \square*

Objects characterized by discrete parameters tend in some sense to be rigid. A deformation of a saturated Lie subalgebra of \mathfrak{sl}_n which remains within the class of such algebras will have a principal element with the same eigenvalues as the original since the eigenvalues change continuously with the algebra and are integers. Those with vanishing Lie (Chevalley-Eilenberg) second cohomology with coefficients in themselves are, of course rigid. This includes the maximal parabolic subalgebras, whether or not they are Frobenius. It would be interesting

to have examples of finite dimensional Frobenius Lie algebras which are not rigid (if such exist).

5. SMALL FUNCTIONALS

Although the eigenvalues of a saturated Frobenius subalgebra $\mathfrak{f} \subset \mathfrak{sl}_n$ depend only on \mathfrak{f} , at this writing we know no way to compute them without first computing a principal element. The generic functional on a Frobenius Lie algebra is Frobenius but the computation is simpler when we have a “small” Frobenius functional. From any set S of pairs of integers (i, j) , $i \neq j$, $1 \leq i, j \leq n$ one can construct both a functional $F_S = \sum_{(i,j) \in S} e_{ij}^*$ on all of \mathfrak{sl}_n (which can then be restricted to any subalgebra) and a directed graph Γ_S whose vertices are the integers $1, \dots, n$ with a directed edge from i to j whenever $(i, j) \in S$. Call a functional F *small* if it has the form F_S with $\#S = n - 1$ and if Γ_S is connected. The last condition is unnecessary by the same argument as in the previous section whenever F_S is Frobenius on some saturated $\mathfrak{f} \subset \mathfrak{sl}_n$. It implies, in particular, that as an undirected graph Γ_S is a tree, i.e., contains no loops. The Dergachev-Kirillov [1] functionals on seaweed algebras (as well as those introduced in [6]) are small.

Suppose that S is a set of $n - 1$ pairs of indices (i, j) ; $i, j \leq n$ such that the graph Γ_S is a tree. (This implies that $i \neq j$, else there were a loop at i , and that every $i \in \{1, \dots, n\}$ is a vertex of Γ_S .) For any $s = (i, j) \in S$ let e_s denote the matrix unit e_{ij} . We now define diagonal matrices d_s , $s \in S$ such that $[d_s, e_s] = e_s$ while $[d_s, e_{s'}] = 0$ for $s \neq s'$. First set $\varepsilon_i = e_{ii} - (1/n) \cdot 1$, (here, 1 represents the $n \times n$ identity matrix. Since $\sum_{i=1}^n \varepsilon_i = 0$ we need only the first $n - 1$ of the ε_i but it is convenient to have all. If $s = (i, j) \in S$ then removing the arrow from i to j disconnects Γ_S . Every $k \in \{1, \dots, n\}$ remains connected either to i or to j . Set d_s equal to the sum of all those ε_k with k still connected to i . (If n remains connected to i then this sum will involve ε_n , in which case one could replace it by the negative of the sum of all $\varepsilon_{k'}$ where k' remains connected to j .)

Lemma 2. *The d_s are linearly independent.*

PROOF. The directed graph Γ_S defines a partial order on the set $\{1, \dots, n\}$. Conjugating by a suitable permutation matrix we may assume that n is a terminal vertex of Γ_S and that the partial order is compatible with the natural order. The d_s , which can now simply be numbered as d_1, \dots, d_{n-1} , by their construction have the property that each d_i is a linear combination only of ε_j with $j \leq i$, with the coefficient of ε_i equal to 1. \square

It follows that the $d_s, s \in S$ span the Cartan subalgebra \mathfrak{h} of diagonal traceless matrices of \mathfrak{sl}_n . Set $D_S = \sum_{s \in S} d_s$.

Theorem 5. *Suppose that $\mathfrak{h} \subset \mathfrak{f} \subset \mathfrak{sl}_n$ and that \mathfrak{f} has a small Frobenius functional F_S . Then $D_S = \widehat{F}_S$.*

PROOF. Since $\mathfrak{h} \subset \mathfrak{f}$, \mathfrak{f} is saturated. Were the assertion false, $D_S - \widehat{F}_S$ would be in the kernel of B_{F_S} . \square

The hypothesis that $\mathfrak{h} \subset \mathfrak{f}$ seems unnecessarily strong, but it holds, for example, for all Frobenius seaweed algebras since for F_S we can take the Dergachev-Kirillov form. It implies, in particular, that the eigenspaces \mathfrak{f}_m are spanned by those e_{ij} which they contain together, in the case $m = 0$, with the elements of \mathfrak{h} . While the e_{ij} with $(i, j) \in S$ are all in \mathfrak{f}_1 they generally do not span that eigenspace. To find the other e_{ij} in \mathfrak{f}_1 and in fact the entire eigenspace decomposition not only of \mathfrak{f} but of all of \mathfrak{sl}_n , suppose that $i, j \in \{1, \dots, n\}, i \neq j$. There is then a path in Γ_S from i to j whose *weight* m we define to be the number of directed edges traversed in the positive direction going from i to j minus the number traversed in the negative direction. Then (with the notation of the preceding paragraph) e_{ij} will be an eigenvector of $\text{ad } D_S$ with eigenvalue m , so if $e_{ij} \in \mathfrak{f}$ then $e_{ij} \in \mathfrak{f}_m$. In the cases examined so far (with the above hypothesis) one finds that in the decomposition $\mathfrak{f} = \mathfrak{f}_- + \mathfrak{f}_0 + \mathfrak{f}_+$ the positive eigenspace \mathfrak{f}_+ is generated by \mathfrak{f}_1 and \mathfrak{f}_- is generated by \mathfrak{f}_{-1} ; we suspect this is true in general. Note that if $e_{ij}, i \neq j$ has weight m then its transpose e_{ji} has weight $-m$, but if $e_{ij} \in \mathfrak{f}$ and m is the highest weight in \mathfrak{f} then $e_{ji} \notin \mathfrak{f}$ since the dual space to \mathfrak{f}_{-m} would be \mathfrak{f}_{1+m} ; this of course holds more generally.

While the eigenvalues of $\text{ad } D_S$ on \mathfrak{f} above have an intrinsic meaning, it is unlikely that this is so for those of its summands d_s since when \mathfrak{f} is Frobenius, its small Frobenius functionals need not all be congruent under the action of SL_n on \mathfrak{f}^* . However, the generic functional on the eigenspace \mathfrak{f}_1 (extended to be zero on all other eigenspaces) is certainly also a Frobenius functional and it may be that these are all congruent. The Dergachev-Kirillov functionals on seaweed algebras are small index functionals and the corresponding D has, as observed, the property that $\text{ad } D$ has integer eigenvalues. It may be that even for seaweed algebras which are not Frobenius these eigenvalues also depend only on the algebra.

For the first parabolic subalgebra of \mathfrak{sl}_n we know from [5] that $F = e_{12}^* + e_{23}^* + \dots + e_{n,n-1}^*$ will serve as a (small) Frobenius functional. For this F one finds from the preceding that $\widehat{F} = (1/2)[(n-1)e_{11} + (n-3)e_{22} + \dots + (1-n)e_{nn}]$, which is just half of the semisimple element of

the principal three-dimensional subalgebra of \mathfrak{sl}_n defined by Kostant in [9]. (The scaling is, of course, immaterial.) The eigenvalues of $\text{ad } \widehat{F}$ in this very special case are $n-1, n-2, \dots, 2-n$. In general, the principal element of \mathfrak{f} is not the semisimple element of any three-dimensional subalgebra of \mathfrak{f} because of the presence of repeated eigenvalues. However, the principal three-dimensional subalgebra is defined for any simple Lie algebra, raising the question of whether this is just a coincidence or whether the principal three-dimensional subalgebra of an arbitrary simple \mathfrak{g} is always related to some Frobenius parabolic subalgebra as it is in \mathfrak{sl}_n .

6. REMARKS

The hypothesis that the coefficient field is \mathbb{C} is convenient but can obviously be replaced simply by the assumption that it is of characteristic zero. Since the principal element of a Frobenius Lie algebra remains well-defined even in characteristic p the results above should continue to hold with some restrictions on the characteristic even in that case, but for the proof we will probably need a better understanding of the meaning of the eigenvalues. While they can be computed by choosing an arbitrary Frobenius functional, the arbitrariness suggests that they also have some other meaning and should be computable directly from the algebra. Some more explicit computations are given in [6]. For a small Frobenius form F_S the only fraction involved in the computation of \widehat{F} is $1/n$. It is conceivable (but would be quite remarkable and strongly reminiscent of Maschke's theorem) if all one really required of the coefficient ring was that n be invertible. Finally, we have assumed here that the Frobenius Lie algebra \mathfrak{f} is a saturated subalgebra of \mathfrak{sl}_n but as noted above, Dergachev has concluded that similar results, including integrality of the eigenvalues and their independence of the choice of Frobenius form F continue to hold with \mathfrak{sl}_n replaced by any simple finite-dimensional Lie algebra \mathfrak{g} .

Elashvili has kindly communicated to us some historical notes. The name "Frobenius Lie algebra" was suggested (in analogy with the associative case) by G. B. Seligman, a colleague of N. Jacobson who was Ooms' thesis adviser (Yale University, 1972). Ooms [10], [11] answered the following question raised by Jacobson, What are the necessary and sufficient conditions on a Lie algebra \mathfrak{g} in order that its enveloping algebra $U(\mathfrak{g})$ be primitive? He showed, in particular, that if L is Frobenius then $U(\mathfrak{g})$ is primitive, and the converse holds if \mathfrak{g} is algebraic. He studied the structure of these algebras in [12] using (without the name) what are called here principal elements and showed that if \widehat{F}

and \widehat{F}' are principal elements of \mathfrak{g} then there exists a g in the adjoint algebraic group G such that $\widehat{F}' = g(\widehat{F})$. (That is, G is the smallest algebraic subgroup of $\text{Aut}(\mathfrak{g})$ such that its Lie algebra contains $\text{ad } \mathfrak{g}$.) The Lie-theoretic properties of \widehat{F} therefore depend only on \mathfrak{g} , which implies the present Theorem 3. In the same paper Ooms shows by example that in general the eigenvalues of a principal element need not be integers.

Elashvili acknowledges the influence of Ooms in his own work on the index of Lie algebras, cf. [2], [3], and [4]. In the first of these (which unfortunately still exist only in preprint form) he proved that the i th maximal parabolic subalgebra of $\mathfrak{sl}(n)$ is Frobenius if and only if i is relatively prime to n . Elashvili also refers to an extensive paper of M. Rais [14] which amongst other results, shows the following. Denote by $G_{n,p}$ the semidirect product $M_{n,p} \times \text{GL}(n)$, where $M_{n,p}$ is the space of $n \times p$ matrices, and by \mathfrak{g} its Lie algebra. Then \mathfrak{g} is Frobenius if and only if p divides n . There has been much recent work on Frobenius Lie algebras, which holds hope of leading to a classification.

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